STATISTICS II

Lisbon School of Economics & Management
Universidade de Lisboa

Bachelor's degrees in Economics, Finance and Management

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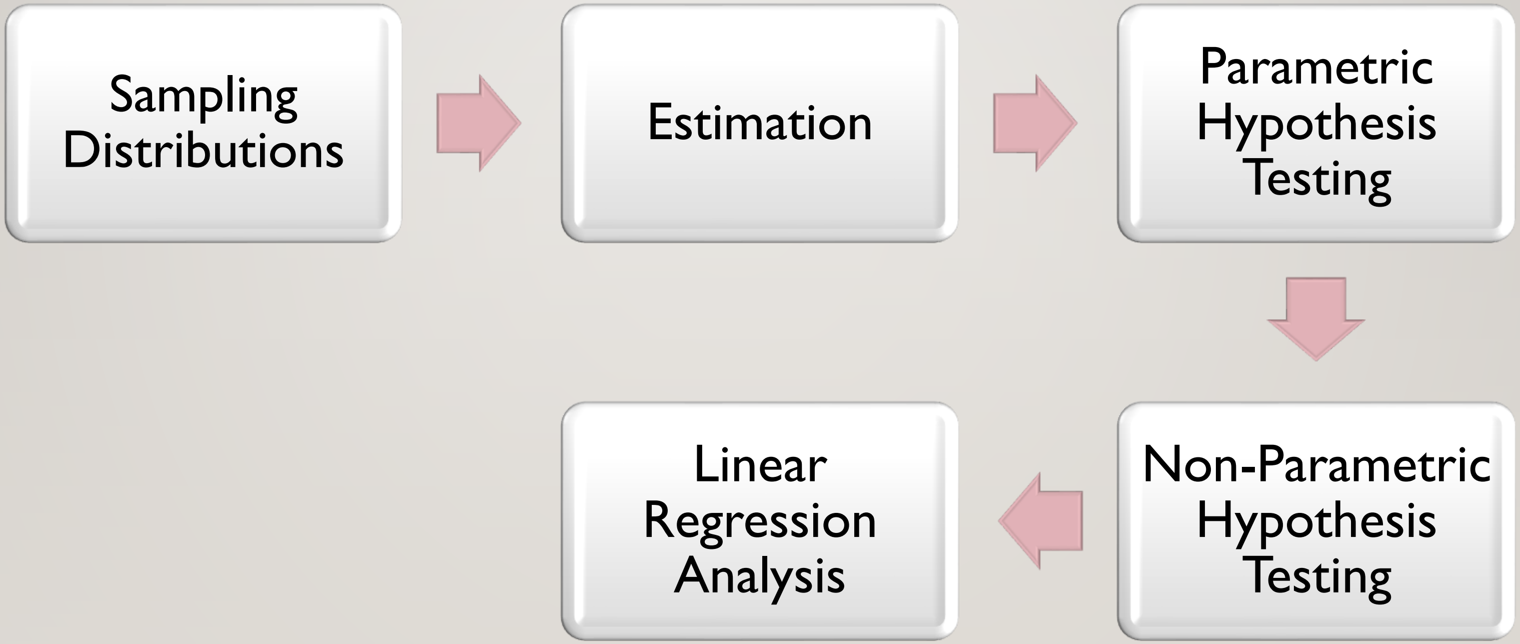
https://doity.com.br/estatistica-aplicada-a-nutricao

https://basiccode.com.br/produto/informatica-basica/

PROGRAM


Sampling Distributions
Estimation
Parametric Hypothesis Testing
Non-Parametric Hypothesis Testing
Linear Regression Analysis

LECTURE 10: CONFIDENCE INTERVAL FOR THE DIFFERENCE BETWEEN TWO POPULATION PROPORTIONS

CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN TWO PROPORTIONS

Population proportions

Goal: Form a confidence interval for the difference between two population proportions, $p_1 - p_2$

Assumptions:

Both sample sizes are large (generally at least 40 observations in each sample)

The point estimate for the difference is $\hat{p}_1 - \hat{p}_2$

CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN TWO PROPORTIONS

Population proportions

- The random variable

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\dfrac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \dfrac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

is approximately normally distributed

CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN TWO PROPORTIONS

Population proportions

The confidence limits for

$p_1 - p_2$ are :

$$\left(\hat{p}_1 - \hat{p}_2\right) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\left(1-\hat{p}_1\right)}{n_1} + \frac{\hat{p}_2\left(1-\hat{p}_2\right)}{n_2}}$$

CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN TWO PROPORTIONS: EXAMPLE

Form a 90% confidence interval for the difference between the proportion of men and the proportion of women who have college degrees.

- In a random sample, 26 of 50 men and 28 of 40 women had an earned college degree

CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN TWO PROPORTIONS: EXAMPLE

Men: $\hat{p}_1 = \frac{26}{50} = 0.52$

Women: $\hat{p}_2 = \frac{28}{40} = 0.70$

$$\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} = \sqrt{\frac{0.52(0.48)}{50} + \frac{0.70(0.30)}{40}} = 0.1012$$

For 90% confidence, $1 - \alpha = 0.9 \Rightarrow \alpha = 0.1 \Rightarrow 1 - \alpha/2 = 0.95 \Rightarrow z_{0.95} = 1.645$

Standard Normal Distribution Table

CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN TWO PROPORTIONS: EXAMPLE

The confidence limits are:

$$\left(\hat{p}_1 - \hat{p}_2\right) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\left(1-\hat{p}_1\right)}{n_1} + \frac{\hat{p}_2\left(1-\hat{p}_2\right)}{n_2}}$$

$$= \left(.52 - .70\right) \pm 1.645\left(0.1012\right)$$

so the confidence interval is

$CI_{90\%}$ ($\mathbf{p_1 - p_2}$) = (-0.3465, -0.0135)

$$-0.3465 < p_1 - p_2 < -0.0135$$

Since this interval does not contain zero we are 90% confident that the two proportions are not equal

EXERCISE 8.20

8.20 In a random sample of 120 large retailers, 85 used regression as a method of forecasting. In an independent random sample of 163 small retailers, 78 used regression as a method of forecasting. Find a 98% confidence interval for the difference between the two population proportions.

Newbold et al (2013)

EXERCISE 8.20: SOLUTION

$$CI_{(1-\alpha)}(\mathbf{p_1}-\mathbf{p_2}) = \left((\hat{p}_1-\hat{p}_2) - z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}, (\hat{p}_1-\hat{p}_2) + z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \right)$$

Answer:

Let

p_1 = population proportion of **large retailers** using regression

p_2 = population proportion of **small retailers** using regression

From the samples:

$$\hat{p}_1 = \frac{85}{120} = 0.7083, \qquad \hat{p}_2 = \frac{78}{163} = 0.4785$$

The point estimate of the difference is:

$$\hat{p}_1 - \hat{p}_2 = 0.7083 - 0.4785 = 0.2298$$

Since both sample sizes are large, we use the **normal approximation**.

The standard error is:

$$SE = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{120} + \frac{\hat{p}_2(1-\hat{p}_2)}{163}}$$

$$SE = \sqrt{\frac{0.7083(0.2917)}{120} + \frac{0.4785(0.5215)}{163}} \approx 0.0571$$

EXERCISE 8.20: SOLUTION

Answer:

Standard Normal Distribution Table

For a **98% confidence interval**, the critical value is:

$$z_{0.99} = 2.326$$

$1 - \alpha = 0.98 \Rightarrow \alpha = 0.02 \Rightarrow 1 - \alpha/2 = 0.98 \Rightarrow z_{0.99} = 2.326$

The margin of error is:

$$ME = z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$ME = 2.326 \times 0.0571 \approx 0.1328$$

98% Confidence Interval

$$(0.2298 - 0.1328,\ 0.2298 + 0.1328)$$

$$\boxed{(0.097,\ 0.363)}$$

$CI_{98\%}(p_1 - p_2) = (0.097, 0.363)$

Note: We are 98% confidente that the proportion of large retailers using regression exceeds that of small retailers by between 9.7% aand 36.3%.

LECTURE 10: CONFIDENCE INTERVAL FOR THE RATIO OF TWO POPULATION VARIANCES

CONFIDENCE INTERVAL FOR THE RATIO OF TWO POPULATION VARIANCES

Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ be two independent random samples of sizes n_1 and n_2, drawn from two normal populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively.

The pivotal quantity is

$$F = \frac{S_1^2}{S_2^2}\,\frac{\sigma_2^2}{\sigma_1^2} \sim F_{n_1-1,\, n_2-1}.$$

Note: The variable F has a Snedecor's F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

The confidence interval can be obtained as follows:

$$P\left(F_{\alpha/2;\, n_1-1,\, n_2-1} < \frac{S_1^2}{S_2^2}\,\frac{\sigma_2^2}{\sigma_1^2} < F_{1-\alpha/2;\, n_1-1,\, n_2-1}\right) = 1 - \alpha$$

$$\Longleftrightarrow \; P\left(\frac{S_1^2}{S_2^2}\,\frac{1}{F_{1-\alpha/2;\, n_1-1,\, n_2-1}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2}\,\frac{1}{F_{\alpha/2;\, n_1-1,\, n_2-1}}\right) = 1 - \alpha.$$

Here $F_{\alpha/2;\, n_1-1,\, n_2-1}$ and $F_{1-\alpha/2;\, n_1-1,\, n_2-1}$ denote the quantiles with probabilities $\alpha/2$ and $1-\alpha/2$, respectively, of the $F_{n_1-1,\, n_2-1}$ distribution.

ProbabilidadesEstatistica2019.pdf

CONFIDENCE INTERVAL FOR THE RATIO OF TWO POPULATION VARIANCES

With this notation, when the populations are normal, the $100(1-\alpha)\%$ **confidence interval** for

$$CI_{(1-\alpha)}(\sigma_1^2/\sigma_2^2) = \left(\frac{S_1^2}{S_2^2} \times \frac{1}{F_{1-\frac{\alpha}{2};\, n_1-1, n_2-1}}, \frac{S_1^2}{S_2^2} \times \frac{1}{F_{\frac{\alpha}{2};\, n_1-1, n_2-1}}\right)$$

$$\frac{\sigma_1^2}{\sigma_2^2}$$

is given by

$$\left(\frac{S_1^2}{S_2^2}\,\frac{1}{F_{1-\alpha/2;\, n_1-1,\, n_2-1}}\,,\ \frac{S_1^2}{S_2^2}\,\frac{1}{F_{\alpha/2;\, n_1-1,\, n_2-1}}\right).$$

Since

$$F_{\alpha/2;\, n_1-1,\, n_2-1} = \frac{1}{F_{1-\alpha/2;\, n_2-1,\, n_1-1}},$$

the interval may equivalently be written as

$$\left(\frac{S_1^2}{S_2^2}\,\frac{1}{F_{1-\alpha/2;\, n_1-1,\, n_2-1}}\,,\ \frac{S_1^2}{S_2^2}\,F_{1-\alpha/2;\, n_2-1,\, n_1-1}\right).$$

ProbabilidadesEstatistica2019.pdf

EXERCISE 37 A)

37. To compare two teaching methods, a class of 22 students was randomly divided into two equal groups. Each group was taught using a different method (Method 1 and Method 2) and, at the end of the course, all students took the same assessment test. The test scores, on a scale from 0 to 100, were as follows:

- **Method 1 (Group 1):** $\bar{x}_1 = 74.8, \quad s_1^2 = 81.5$
- **Method 2 (Group 2):** $\bar{x}_2 = 72.1, \quad s_2^2 = 110.5$

Assume that the scores obtained in each group follow a normal distribution.

a) Construct a **95% confidence interval** for the **ratio of the variances** of the two groups.

b) Taking into account the result from part (a), construct a **99% confidence interval** for the **difference between the means**. Interpret the result obtained.

Murteira et al (2015), Chapter 7

EXERCISE 37 A): SOLUTION

✅ Answer:

We are asked to construct a **95% confidence interval** for the **ratio of the population variances**

$$\frac{\sigma_1^2}{\sigma_2^2}.$$

Given data

The class of 22 students was divided into two equal groups, so:

- Group 1: $n_1 = 11, \quad s_1^2 = 81.5$
- Group 2: $n_2 = 11, \quad s_2^2 = 110.5$

Assume both populations are **normally distributed**.

Step 1: Sampling distribution

For normal populations, the pivotal quantity

$$F = \frac{S_1^2}{S_2^2}\frac{\sigma_2^2}{\sigma_1^2}$$

follows a **Snedecor F distribution** with

$$(n_1 - 1,\ n_2 - 1) = (10,\ 10)$$

degrees of freedom.

Note: The variable F has a Snedecor's F distribution with 10 and 10 degrees of freedom.

EXERCISE 37 A): SOLUTION

✅ Answer:

$$CI_{(1-\alpha)}(\sigma_1^2/\sigma_2^2) = \left(\frac{S_1^2}{S_2^2} \times \frac{1}{F_{1-\frac{\alpha}{2};\, n_1-1, n_2-1}}, \frac{S_1^2}{S_2^2} \times \frac{1}{F_{\frac{\alpha}{2};\, n_1-1, n_2-1}}\right)$$

Step 2: Confidence interval formula

A $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is

$$\left(\frac{s_1^2}{s_2^2}\frac{1}{F_{1-\alpha/2;\,10,10}}, \frac{s_1^2}{s_2^2}\frac{1}{F_{\alpha/2;\,10,10}}\right).$$

For a **95% confidence level**, $\alpha = 0.05$.

Step 3: Numerical values

$$\frac{s_1^2}{s_2^2} = \frac{81.5}{110.5} = 0.737$$

From the F distribution with $(10, 10)$ degrees of freedom:

$$F_{1-0.05/2;\,10,10} = F_{0.975;\,10,10} \approx 3.72$$

$$F_{0.025;\,10,10} \approx \frac{1}{3.72} = 0.27$$

1 – α = 0.95 (confidence level), then
$F_{1-\frac{\alpha}{2};\, n_1-1, n_2-1} = F_{0.975;10,10} = 3.72$ and
$F_{\frac{\alpha}{2};\, n_1-1, n_2-1} = F_{0.025;10,10} = 1/3.72 = 0.27$
(see F Table)

EXERCISE 37 A): SOLUTION

$F_{0.975;10,10} = F_{0.025;10,10}{}^{*} = 3.72$ and
$F_{0.025;10,10} = 1/\ F_{0.975;10,10} = 1/3.72 = 0.27$

Answer:

$$F_{m,n,\varepsilon} : P(X > F_{m,n,\varepsilon}) = \varepsilon$$

n – graus de liberdade do denominador

		m – graus de liberdade do numerador																		
n	ε	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
10	.100	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
	.050	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
	.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
	.010	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	.100	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
	.050	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
	.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88
	.010	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	.100	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
	.050	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
	.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
	.010	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	.100	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
	.050	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
	.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60
	.010	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	.100	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
	.050	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
	.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49
	.010	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	.100	3.07	2.70	2.49	2.36	2.27	2.21	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.050	4.54	3.68	3.29	3.06	2.90	2.79	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.025	6.20	4.77	4.15	3.80	3.58	3.41	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.010	8.68	6.36	5.42	4.89	4.56	4.32	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
16	.100	3.05	2.67	2.46	2.33	2.24	2.18	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.050	4.49	3.63	3.24	3.01	2.85	2.74	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.025	6.12	4.69	4.08	3.73	3.50	3.34	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.010	8.53	6.23	5.29	4.77	4.44	4.20	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
17	.100	3.03	2.64	2.44	2.31	2.22	2.15	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.050	4.45	3.59	3.20	2.96	2.81	2.70	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.025	6.04	4.62	4.01	3.66	3.44	3.28	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.010	8.40	6.11	5.19	4.67	4.34	4.10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
18	.100	3.01	2.62	2.42	2.29	2.20	2.13	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	.050	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
	.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
	.010	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	.100	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
	.050	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
	.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
	.010	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49

Note on F quantiles:

- $F_{1-\alpha}$ denotes the **quantile of the F distribution with $1-\alpha$ probability on the left**, which corresponds to a **right-tail area of α**.
- F^*_α denotes the **same quantile**, but with the **area α on the right** (as usually found in tables).
- Without the asterisk, F_α refers to the **left-tail area**; with the asterisk, F^*_α refers to the **right-tail area**, matching the conventions used in most F distribution tables.

EXERCISE 37 A): SOLUTION

Answer:

Step 4: Confidence interval

$$\text{Lower bound} = 0.737/3.72 = 0.198$$

$$\text{Upper bound} = 0.737/0.27 = 2.73$$

Final Answer

A **95% confidence interval** for the ratio of the population variances

$$\frac{\sigma_1^2}{\sigma_2^2}$$

is:

$CI_{95\%}$ (**σ_1^2 / σ_2^2**) = (0.198, 2.73)

Note: The interval contains the value 1, so there is **no evidence of a difference between the population variances** at the 5% significance level.

EXERCISE 36 C)

An agricultural company operates two identical tractors. The fuel consumption per hour of operation for each tractor is a random variable with a normal distribution and unknown parameters. For the purpose of monitoring fuel consumption, the following random samples were obtained for the two tractors:

- **Tractor 1:** 9.0, 9.5, 9.8, 9.4, 10.0, 10.2, 9.6, 9.7, 9.5
- **Tractor 2:** 10.0, 9.6, 9.9, 9.7, 10.1

Determine a **90% confidence interval** for the **ratio of the variances** of the fuel consumption of the two tractors.

Murteira et al (2015), Chapter 7

EXERCISE 36 C): SOLUTION

Answer:

We are asked to construct a **90% confidence interval** for the ratio of the population variances

$$\frac{\sigma_1^2}{\sigma_2^2},$$

assuming **normal distributions with unknown parameters**.

Step 1: Sample sizes

- Tractor 1:

  $n_1 = 9$
- Tractor 2:

  $n_2 = 5$

Step 2: Sample means and variances

Tractor 1 data

$9.0,\ 9.5,\ 9.8,\ 9.4,\ 10.0,\ 10.2,\ 9.6,\ 9.7,\ 9.5$

Sample mean:

$$\bar{x}_1 = 9.633$$

Sample variance:

$$s_1^2 = 0.145$$

Tractor 2 data

$10.0,\ 9.6,\ 9.9,\ 9.7,\ 10.1$

Sample mean:

$$\bar{x}_2 = 9.860$$

Sample variance:

$$s_2^2 = 0.043$$

EXERCISE 36 C): SOLUTION

✅ Answer:

Step 3: Pivotal quantity

For normal populations, the pivotal quantity

$$F = \frac{S_1^2}{S_2^2}\frac{\sigma_2^2}{\sigma_1^2}$$

follows an F **distribution** with:

$$(n_1 - 1,\ n_2 - 1) = (8,\ 4)$$

degrees of freedom.

Step 4: Confidence interval formula

A $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is:

$$\left(\frac{s_1^2}{s_2^2}\frac{1}{F_{1-\alpha/2;\,8,4}},\ \frac{s_1^2}{s_2^2}\frac{1}{F_{\alpha/2;\,8,4}}\right).$$

Here, $\alpha = 0.10$.

Step 5: Numerical values

Ratio of sample variances:

$$\frac{s_1^2}{s_2^2} = \frac{0.145}{0.043} = 3.37$$

From the F distribution:

$$F_{1-\alpha/2;\,8,4} = F_{0.95;\,8,4} \approx 6.04$$

$$F_{\alpha/2;\,8,4} = F_{0.05;\,8,4} \approx \frac{1}{9.55} = 0.105$$

1 – α = 0.90 (confidence level), then
$F_{1-\frac{\alpha}{2}; n_1-1, n_2-1} = F_{0.95;8,4}$ = 6.04 and
$F_{\frac{\alpha}{2}; n_1-1, n_2-1} = F_{0.05;8,4} = 1/F_{0.95;4,8}$ = 1/3.84 = 0.26
(see F Table)

EXERCISE 36 C): SOLUTION

$F_{0.95;8,4} = F^*_{0.05;8,4} = 6.04$ and
$F_{0.05;8,4} = F^*_{0.95;8,4} = 1/\ F^*_{0.05;4,8} = 1/3.84 = 0.26$

Answer:

$$F_{m,n,\varepsilon} : P(X > F_{m,n,\varepsilon}) = \varepsilon$$

n – graus de liberdade do denominador		m - graus de liberdade do numerador																		
	ε	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	.100	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.00	62.26	62.53	62.79	63.06	63.33
	.050	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.90	245.95	248.02	249.05	250.10	251.14	252.20	253.25	254.32
	.025	647.79	799.48	864.15	899.60	921.83	937.11	948.20	956.64	963.28	968.63	976.72	984.87	993.08	997.27	1001.40	1005.60	1009.79	1014.04	1018.26
	.010	4052.18	4999.34	5403.53	5624.26	5763.96	5858.95	5928.33	5980.95	6022.40	6055.93	6106.68	6156.97	6208.66	6234.27	6260.35	6286.43	6312.97	6339.51	6365.59
2	.100	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
	.050	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
	.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
	.010	98.50	99.00	99.16	99.25	99.30	99.33	99.36	99.38	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.48	99.48	99.49	99.50
3	.100	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13
	.050	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
	.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
	.010	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.34	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	.100	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
	.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
	.010	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	.100	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.11
	.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
	.010	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	.100	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
	.050	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
	.010	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	.100	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
	.050	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.41	4.36	4.31	4.25	4.20	4.14
	.010	12.25	9.55	8.45	7.85	7.46														
8	.100	3.46	3.11	2.92	2.81	2.73														
	.050	5.32	4.46	4.07	3.84	3.69														
	.025	7.57	6.06	5.42	5.05	4.82														
	.010	11.26	8.65	7.59	7.01	6.63														
9	.100	3.36	3.01	2.81	2.69	2.61														
	.050	5.12	4.26	3.86	3.63	3.48														
	.025	7.21	5.71	5.08	4.72	4.48														
	.010	10.56	8.02	6.99	6.42	6.06														

Note on F quantiles:

- $F_{1-\alpha}$ denotes the **quantile of the F distribution with $1-\alpha$ probability on the left**, which corresponds to a **right-tail area of α**.
- F^*_α denotes the **same quantile**, but with the **area α on the right** (as usually found in tables).
- Without the asterisk, F_α refers to the **left-tail area**; with the asterisk, F^*_α refers to the **right-tail area**, matching the conventions used in most F distribution tables.

EXERCISE 36 C): SOLUTION

✅ Answer:

Previously computed:

$$\frac{s_1^2}{s_2^2} = 3.37$$

The **90% confidence interval** is:

$$\left(\frac{3.37}{6.04}, \frac{3.37}{0.26}\right) = (0.56,\ 12.96)$$

✅ **Final Answer**

A **90% confidence interval** for the ratio of the population variances

$$\frac{\sigma_1^2}{\sigma_2^2}$$

Note: With 90% confidence, the variance of hourly fuel consumption for Tractor 1 may be between **0.56 and 12.96 times** the variance of Tractor 2.
Since the interval includes 1, there is **no evidence of a significant difference in variances** at the 10% significance level.

$CI_{95\%}\ (\sigma_1^2 / \sigma_2^2) = (0.56, 12.96)$

LECTURE 10 HOMEWORK: QUESTIONS AND SOLUTIONS

EXERCISE 8.24

8.24 Supermarket shoppers were observed and questioned immediately after putting an item in their cart. Of a random sample of 510 choosing a product at the regular price, 320 claimed to check the price before putting the item in their cart. Of an independent random sample of 332 choosing a product at a special price, 200 made this claim. Find a 90% confidence interval for the difference between the two population proportions.

Newbold et al (2013)

EXERCISE 8.24: SOLUTION

Answer:

We are asked to construct a **90% confidence interval** for the difference between two population proportions:

$$p_1 - p_2$$

where:

- **Regular price group**: $n_1 = 510$, $x_1 = 320 \Rightarrow \hat{p}_1 = \frac{320}{510} = 0.6275$
- **Special price group**: $n_2 = 332$, $x_2 = 200 \Rightarrow \hat{p}_2 = \frac{200}{332} = 0.6024$

Step 1: Difference in sample proportions

$$\hat{p}_1 - \hat{p}_2 = 0.6275 - 0.6024 = 0.0251$$

$$CI_{(1-\alpha)}(\mathbf{p_1} - \mathbf{p_2}) = \left((\hat{p}_1 - \hat{p}_2) - z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}, (\hat{p}_1 - \hat{p}_2) + z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \right)$$

EXERCISE 8.24: SOLUTION

✅ Answer:

Step 2: Standard error

For independent samples:

$$SE = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$SE = \sqrt{\frac{0.6275(1-0.6275)}{510} + \frac{0.6024(1-0.6024)}{332}}$$

$$SE = \sqrt{\frac{0.2342}{510} + \frac{0.2395}{332}} = \sqrt{0.000459 + 0.000722} = \sqrt{0.001181} = 0.03437$$

Step 3: Critical value

For a **90% confidence level**, $z_{0.95} = 1.645$.

Step 4: Margin of error

$$ME = 1.645 \times 0.03437 \approx 0.0565$$

Step 5: Confidence interval

$CI_{90\%}$ ($\mathbf{p_1}$ - $\mathbf{p_2}$) = (-0.0314 0.082)

$$\hat{p}_1 - \hat{p}_2 \pm ME = 0.0251 \pm 0.0565$$

$$CI = (-0.0314, 0.0816)$$

Note: Since the interval includes 0, there is **no significant difference** at the 10% significance level between the proportions of shoppers who check the price for regular versus special price items.

LECTURE 11: ESTIMATION METHODS - METHOD OF MOMENTS

ESTIMATION METHODS

Among the most commonly used **point estimation methods**, the following stand out:

- **Method of Moments** – the estimators are obtained by replacing the population moments in the corresponding expressions with the sample moments. Under very general conditions (Murteira et al., 2007), the resulting estimators are consistent and have a Normal distribution when the sample size is large.
- **Least Squares Method** – this method is usually applied in the context of linear regression.
- **Maximum Likelihood Method** – this is probably the most important estimation method. In general, maximum likelihood estimators possess the desirable properties of a good estimator: they are efficient and consistent. Although they are usually not unbiased, they are asymptotically unbiased. Moreover, they have an asymptotic Normal distribution

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METHOD OF MOMENTS

This estimation method is one of the simplest and oldest approaches for obtaining estimators of one or more parameters of a distribution. The basic idea is to use the sample moments to estimate the corresponding population moments and, from these, to estimate the parameters of interest (Murteira et al., 2007).

Let $X_1, X_2, \ldots, X_n$ be a random sample from a given population with probability (density) function $f(x; \theta_1, \theta_2, \ldots, \theta_k)$, which depends on k parameters.
Assuming that the ordinary population moments μ_r of the random variable X exist, these moments are functions of the k parameters and are given by

$$\mu_r = E(X^r) = \begin{cases} \sum_x x^r f(x; \theta_1, \theta_2, \ldots, \theta_k), & \text{for discrete distributions,} \\ \int_{-\infty}^{+\infty} x^r f(x; \theta_1, \theta_2, \ldots, \theta_k)\, dx, & \text{for continuous distributions.} \end{cases}$$

Ordinary Population Moments

The corresponding **sample moments** are given by

Sample Moments

$$m_r = \frac{1}{n} \sum_{i=1}^{n} X_i^r.$$

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METHOD OF MOMENTS

The **method of moments** consists of assuming that the estimators of the ordinary moments are given by the corresponding sample moments, that is,

$$\hat{\mu}_r = m_r, \quad r = 1, \ldots, k.$$

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Note on the method of moments:

- When we need to estimate **only one parameter**, we equate the **first population moment** to the **first sample moment**.
- When we need to estimate **two parameters**, we require **two equations**: we set the **first population moment equal to the first sample moment** and the **second population moment equal to the second sample moment**.
- In general, to estimate k parameters, we equate the **first** k **population moments** to the **first** k **sample moments**, successively.

METHOD OF MOMENTS: EXAMPLE 1

Let $X_1, X_2, \ldots, X_n$ be a random sample from an **Exponential distribution**, $X \sim \mathrm{Exp}(\lambda)$. Estimate the parameter λ using the **method of moments**.

Let $X_1, X_2, \ldots, X_n$ be a random sample from an **Exponential distribution** with parameter λ:

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

1. **Population moment**

The first population moment (mean) of X is:

$$\mu_1 = E[X] = \frac{1}{\lambda}.$$

2. **Sample moment**

The first sample moment is the **sample mean**:

$$m_1 = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

METHOD OF MOMENTS: EXAMPLE 1

3. Method of moments estimator

Set the population moment equal to the sample moment:

$$\mu_1 = m_1 \quad \Rightarrow \quad \frac{1}{\lambda} = \bar{X}.$$

Solve for λ:

$$\hat{\lambda} = \frac{1}{\bar{X}}.$$

✅ **Interpretation:** The method of moments estimator of λ is the reciprocal of the sample mean.

The **method of moments estimator** of the parameter λ of the exponential distribution is

Estimator

$$\hat{\lambda}_{MM} = \frac{1}{\bar{X}},$$

and the corresponding **estimate** from a sample is

Estimate

$$\hat{\lambda}_{MM} = \frac{1}{\bar{x}}.$$

METHOD OF MOMENTS: EXAMPLE 2

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Normal distribution, $X \sim N(\mu, \sigma^2)$. Estimate the parameters μ and σ using the **method of moments**.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a **Normal distribution**:

$$X \sim N(\mu, \sigma^2).$$

We want to estimate the parameters μ and σ using the **method of moments**.

Note: The **corrected variance** uses $n-1$ in the denominator, while the **uncorrected variance** uses n. In general, we usually use the **corrected variance**.

3. Equate population and sample moments

$$\mu_1 = m_1 \quad \Rightarrow \quad \hat{\mu} = \bar{X}.$$

$$\mu_2 = m_2 \quad \Rightarrow \quad \hat{\sigma}^2 + \hat{\mu}^2 = m_2 \quad \Rightarrow \quad \hat{\sigma}^2 = m_2 - \bar{X}^2.$$

Corrected variance ($s^{2'}$)

$$\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}$$

✅ **Method of moments estimators:**

Estimators

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = m_2 - \bar{X}^2.$$

$$\begin{cases} \mu = \dfrac{1}{n}\sum_{i=1}^{n} x_i \\ \sigma^2 + \mu^2 = \dfrac{1}{n}\sum_{i=1}^{n} x_i^2 \end{cases} \Leftrightarrow \begin{cases} \mu = \bar{x} \\ \sigma^2 = \dfrac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 \end{cases}.$$

Uncorrected variance (s^2)

Interpretation:

- The **mean** is estimated by the **sample mean**,
- The **variance** is estimated by the **second sample moment minus the square of the sample mean**.

LECTURE 11: ESTIMATION METHODS – MAXIMUM LIKELIHOOD METHOD

MAXIMUM LIKELIHOOD METHOD (MLM)

This method can only be applied if the population distribution is known.

Definition: Let $X_1, X_2, \ldots, X_n$ be a random sample from a given population with probability (density) function

$$f(x; \theta_1, \theta_2, \ldots, \theta_k) = f(x; \theta).$$

Then the joint probability (density) function of the sample variables is given by:

$$f(x_1, x_2, \ldots, x_n; \theta) = f(x_1; \theta) f(x_2; \theta) \ldots f(x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta).$$

For a given sample, the function of θ is called the **likelihood function**:

$$\mathcal{L}(\theta) = \mathcal{L}(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta).$$

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MAXIMUM LIKELIHOOD METHOD (MLM)

The **maximum likelihood method** consists of finding the estimator $\hat{\theta}$ that maximizes the value of the likelihood function for a given sample; that is, the value of θ that makes the observed sample most probable, i.e., most **likely**.

Frequently, the maximum likelihood estimator can be found by derivation, following these steps:

1. Determine the likelihood function $\mathcal{L}(\theta)$.
2. If necessary, apply the logarithmic transformation to the likelihood function, $\ln(\mathcal{L}(\theta))$. This transformation often simplifies the maximization problem.
3. Find the points where the **first derivative** of the function $\mathcal{L}(\theta)$, or $\ln(\mathcal{L}(\theta))$, with respect to each θ_i vanishes (first-order condition):

$$\frac{\partial \mathcal{L}(\theta)}{\partial \theta_i} = 0 \quad \text{or} \quad \frac{\partial \ln(\mathcal{L}(\theta))}{\partial \theta_i} = 0.$$

4. Verify that the **second derivative** of the function $\mathcal{L}(\theta)$, or $\ln(\mathcal{L}(\theta))$, with respect to θ_i is negative (second-order condition):

$$\frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta_i^2} < 0 \quad \text{or} \quad \frac{\partial^2 \ln(\mathcal{L}(\theta))}{\partial \theta_i^2} < 0.$$

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MAXIMUM LIKELIHOOD METHOD: EXAMPLE 1

Consider the random variable $X \sim Poisson(\lambda)$ and the sample (0, 0, 2, 5, 3, 1). . Determine a **maximum likelihood estimate** of λ.

Step 0: Problem

We have $X \sim \text{Poisson}(\lambda)$ and a sample:

$$0, 0, 2, 5, 3, 1$$

We want the **MLE of** λ.

Step 1: Likelihood function

The **Poisson pmf** is:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

The **likelihood function** for the sample is:

$$L(\lambda) = \prod_{i=1}^{6} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum x_i} e^{-6\lambda}}{\prod x_i!}$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 1

Consider the random variable $X \sim Poisson(\lambda)$ and the sample (0, 0, 2, 5, 3, 1). Determine a **maximum likelihood estimate** of λ.

Step 2: Log-likelihood

Take the natural logarithm:

$$\ell(\lambda) = \ln L(\lambda) = \sum_{i=1}^{6} x_i \ln \lambda - 6\lambda - \sum_{i=1}^{6} \ln(x_i!)$$

$$\ell(\lambda) = (\sum x_i) \ln \lambda - 6\lambda - \sum \ln(x_i!)$$

Compute $\sum x_i$:

$$0 + 0 + 2 + 5 + 3 + 1 = 11$$

So:

$$\ell(\lambda) = 11 \ln \lambda - 6\lambda - \sum \ln(x_i!)$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 1

Consider the random variable $X \sim Poisson(\lambda)$ and the sample (0, 0, 2, 5, 3, 1). Determine a **maximum likelihood estimate** of λ.

1. $y = u^n \Rightarrow y' = nu^{n-1}u'$;
2. $y = c \Rightarrow y' = 0$, onde k é uma constante real;
3. $y = uv \Rightarrow y' = u'v + v'u$
4. $y = \frac{u}{v} \Rightarrow y' = \frac{u'v - v'u}{v^2}$
5. $y = a^u \Rightarrow y' = a^u(\ln a)u', (a > 0, a \neq 1)$
6. $y = e^u \Rightarrow y' = e^u u'$
7. $y = \log_a u \Rightarrow y' = \frac{u'}{u}\log_a e$
8. $y = \ln u \Rightarrow y' = \frac{1}{u}u'$
9. $y = u^v \Rightarrow y' = vu^{v-1}u' + u^v(\ln u)v'$
10. $y = \sin u \Rightarrow y' = u'\cos u$
11. $y = \cos u \Rightarrow y' = -u'\sin u$
12. $y = \tan u \Rightarrow y' = u'\sec^2 u$, desde que $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$;
13. $y = \cot u \Rightarrow y' = -u'\csc^2 u$, desde que $x \neq n\pi, n \in \mathbb{Z}$;

Step 3: First derivative

Differentiate with respect to λ:

$$\frac{d\ell}{d\lambda} = \frac{11}{\lambda} - 6$$

Set derivative to zero:

$$\frac{11}{\hat{\lambda}} - 6 = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{11}{6} \approx 1.833$$

Step 4: Second derivative

$$\frac{d^2\ell}{d\lambda^2} = -\frac{11}{\lambda^2} < 0$$

- Negative confirms **maximum**.

Answer

$$\boxed{\hat{\lambda} = 11/6 \approx 1.833}$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 2

Consider the random variable $X \sim Exp(\lambda)$ and the sample (1.2, 0.5, 3). Determine a **maximum likelihood estimate** of λ.

Step 0: Problem

We have $X \sim \text{Exponential}(\lambda)$ and a sample:

$$1.2, 0.5, 3$$

We want the **maximum likelihood estimate (MLE) of** λ.

Step 1: Likelihood function

The pdf of the exponential distribution is:

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0$$

For the sample, the **likelihood function** is:

$$L(\lambda) = \prod_{i=1}^{3} \lambda e^{-\lambda x_i} = \lambda^3 e^{-\lambda \sum x_i}$$

$$\sum x_i = 1.2 + 0.5 + 3 = 4.7$$

So:

$$L(\lambda) = \lambda^3 e^{-4.7\lambda}$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 2

Consider the random variable $X \sim Exp(\lambda)$ and the sample (1.2, 0.5, 3). Determine a **maximum likelihood estimate** of λ.

Step 2: Log-likelihood

$$\ell(\lambda) = \ln L(\lambda) = 3 \ln \lambda - 4.7\lambda$$

Step 3: First derivative

$$\frac{d\ell}{d\lambda} = \frac{3}{\lambda} - 4.7$$

Set derivative equal to zero:

$$\frac{3}{\hat{\lambda}} - 4.7 = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{3}{4.7} \approx 0.638$$

Step 4: Second derivative

$$\frac{d^2\ell}{d\lambda^2} = -\frac{3}{\lambda^2} < 0$$

- Negative confirms **maximum**.

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 2

Consider the random variable $X \sim Exp(\lambda)$ and the sample (1.2, 0.5, 3). Determine a **maximum likelihood estimate** of λ.

Answer

$$\boxed{\hat{\lambda} \approx 0.638}$$

Shortcut: For an exponential distribution, the MLE of λ is always:

$$\hat{\lambda} = \frac{1}{\bar{X}} = \frac{1}{4.7/3} = \frac{3}{4.7} \approx 0.638$$

Note:

In this case, the **maximum likelihood estimator (MLE) of λ** for the exponential distribution is equal to the **method of moments estimator**:

$$\hat{\lambda}_{\text{MLE}} = \hat{\lambda}_{\text{MM}} = \frac{1}{\bar{X}}$$

However, this **coincidence does not always occur** for other distributions or parameters — in general, the MLE and the method of moments estimator can be different.

INVARIANCE PROPERTY OF MAXIMUM LIKELIHOOD ESTIMATORS

The **invariance property** states that if $\hat{\theta}$ is the **maximum likelihood estimator (MLE)** of a parameter θ, then the MLE of any function of that parameter, $g(\theta)$, is simply the same function applied to the MLE of θ.

In other words, if

$$\boxed{\hat{\theta}}$$

is the MLE of θ, then

$$\boxed{g(\hat{\theta})}$$

is the MLE of $g(\theta)$.

$\dot{\theta}\ is\ MLE\ of\ \theta$ ➡ $g(\dot{\theta})\ is\ MLE\ of\, g(\theta)$

Example

If $\hat{\theta}$ is the MLE of θ, and we want to estimate θ^2, then the MLE of θ^2 is

$$\hat{\theta}^2$$

This property simplifies estimation because we do not need to derive a new likelihood function for transformations of the parameter.

INVARIANCE PROPERTY OF MAXIMUM LIKELIHOOD ESTIMATORS: EXAMPLES

Example 1: Exponential distribution

- Let $X_1, \ldots, X_n \sim \text{Exponential}(\lambda)$.
- The MLE of λ is $\hat{\lambda} = 1/\bar{X}$.

Now suppose we want the MLE of the **mean** of the distribution, which is $\mu = 1/\lambda$.

- By invariance:

$$\hat{\mu} = \frac{1}{\hat{\lambda}} = \bar{X}.$$

✅ The MLE of the mean is simply the sample mean.

Example 2: Normal distribution

- Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.
- The MLE of μ is $\hat{\mu} = \bar{X}$.
- The MLE of σ^2 is $\hat{\sigma}^2 = \frac{1}{n} \sum (X_i - \bar{X})^2$.

Now suppose we want the MLE of the **standard deviation**, $\sigma = \sqrt{\sigma^2}$.

- By invariance:

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n} \sum (X_i - \bar{X})^2}.$$

✅ The MLE of the standard deviation is the square root of the MLE of the variance.

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COMPARISON BETWEEN THE METHOD OF MOMENTS AND MAXIMUM LIKELIHOOD ESTIMATORS

Method	Method of Moments (MM)	Maximum Likelihood Method (MLM)
Basic idea	Equate sample moments with population moments	Maximize the likelihood function
Computation	Usually simpler	Often more complex
Statistical properties	May not always be efficient	Often efficient and consistent
Use in practice	Useful for quick estimation	Very widely used in statistics
Invariance property	Does not generally satisfy invariance	Satisfies the invariance property

LECTURE 11 HOMEWORK: QUESTIONS

EXERCISE 1 A), B) AND C)

Let X be a random variable with probability function

$$f(x \mid \theta) = \theta(1-\theta)^x, \quad x = 0, 1, 2, 3, \ldots, \quad 0 < \theta < 1.$$

It is known that $E(X) = \frac{1-\theta}{\theta}$. From a random sample of size $n = 1000$, the following value was observed:

$$\sum_{i=1}^{1000} X_i = 980.$$

a) Obtain an estimate of θ using the **method of moments**.

b) Determine the **maximum likelihood estimator** of θ.

c) Compute, with justification, the **maximum likelihood estimate of the population mean**.

d) Reparametrize the distribution in terms of $\mu = E(X)$, and use the new probability function to estimate the **population mean**.

Murteira (2015), Chapter 7

EXERCISE 5

Consider a random sample of size n drawn from a population with probability density function

$$f(x \mid \theta) = \frac{1}{2\theta}, \quad (-\theta < x < \theta), \quad \text{for } \theta > 0.$$

Compute an estimator for θ using the **method of moments**.

Murteira (2015), Chapter 7

EXERCISE 1: MLM

Let $X_1, \dots, X_n$ be a random sample from a Normal distribution with parameters μ and σ. Estimate the parameters using the **maximum likelihood method**.

THANKS!

Questions?